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FINITE ELEMENT SOLUTION OF THE NAVIER–STOKES EQUATIONS BY A VELOCITY–VORTICITY METHOD

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SUMMARY

A velocity–vorticity formulation of the Navier–Stokes equations is presented as an alternative to the primitive variables approach. The velocity components and the vorticity are solved for in a fully coupled manner using a Newton method. No artificial viscosity is required in this formulation. The pressure is updated by a method allowing natural imposition of boundary conditions. Incompressible and subsonic results are presented for two-dimensional laminar internal flows up to high Reynolds numbers.

KEY WORDS Finite elements Navier–Stokes Velocity–vorticity

INTRODUCTION

Streamfunction–vorticity formulations of the incompressible and compressible Navier–Stokes equations have been successfully implemented for 2D internal and external flows.[1–4] However, in 3D the boundary conditions are complex to implement, particularly for internal flows. The velocity–vorticity formulation has therefore become an increasingly attractive alternative.[5–16] Fasel[5] appears to be among the first to consider the use of these variables for two-dimensional enclosed flows, while Dennis *et al.*[6] proposed a similar formulation for the three-dimensional Navier–Stokes equations.

Among the advantages of the velocity–vorticity approach, one should mention:

(1) Boundary condition implementation is simpler for second-order equations.
(2) For incompressible flows non-inertial effects do not change the form of the equations.[14]
(3) For incompressible flows the pressure does not need to be solved for as one of the variables.

While most approaches focus on finite differences schemes and show results for incompressible internal flows,[5–12,16] the present paper proposes a finite element approach and, in addition, the formulation is extended to subsonic flows. The variables are solved for in a fully coupled manner by a Newton method, achieving very rapid convergence. The geometries analysed include a driven cavity, a trough and a converging–diverging nozzle.

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GOVERNING EQUATIONS

Velocity–vorticity equations

The governing equations for two-dimensional, steady, compressible flow in terms of the primitive variables (u, v, p) are

$$(\rho u)_x + (\rho v)_y = \nabla\cdot(\rho \mathbf{V}) = 0\,, \tag{1}$$

$$Re[\rho u u_x + \rho v u_y] = -Re(p_x) + [2\mu u_x - \tfrac{2}{3}\mu\nabla\cdot\mathbf{V}]_x + [\mu u_y + \mu v_x]_y\,, \tag{2}$$

$$Re[\rho u v_x + \rho v v_y] = -Re(p_y) + [2\mu v_y - \tfrac{2}{3}\mu\nabla\cdot\mathbf{V}]_y + [\mu u_y + \mu v_x]_x\,, \tag{3}$$

where Re is the Reynolds number.

The velocity–vorticity system of equations is obtained by manipulating the vorticity definition ($\mathbf{\Omega} \equiv \nabla \times \mathbf{V}$, with $\mathbf{\Omega} = \omega\mathbf{k}$ in 2D) and the first-order continuity equation as follows:

$$\begin{aligned}\nabla \times \mathbf{\Omega} &= \nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla\cdot\mathbf{V}) - \nabla^2\mathbf{V}\\ &= \nabla\left(\frac{1}{\rho}\nabla\cdot(\rho\mathbf{V}) - \frac{1}{\rho}\mathbf{V}\cdot\nabla\rho\right) - \nabla^2\mathbf{V}\\ &= -\nabla\left(\frac{1}{\rho}\mathbf{V}\cdot\nabla\rho\right) - \nabla^2\mathbf{V}\,.\end{aligned}$$

For two-dimensional flows the equations become

$$\nabla^2 u + \omega_y + \left(\frac{u\rho_x + v\rho_y}{\rho}\right)_x = 0, \tag{4}$$

$$\nabla^2 v + \omega_x + \left(\frac{u\rho_x + v\rho_y}{\rho}\right)_y = 0. \tag{5}$$

In the derivation of these equations, while it has been implicitly recognized that the gradient of the continuity equation is zero, the continuity equation itself is no longer part of the system. Mass continuity is thus accounted for only to within an arbitrary constant.[12] This problem is dealt with in the boundary conditions section.

The vorticity transport equation is obtained by taking the curl of the momentum equations and eliminating the pressure term. After rearranging,

$$\nabla^2(\mu\omega) - Re(\rho u\omega_x + \rho v\omega_y + S^\rho) + (S^{\mu u})_x + (S^{\mu v})_y = 0, \tag{6}$$

where

$$S^\rho = u(\rho_x u_y - \rho_y u_x) + v(\rho_x v_y - \rho_y v_x)\,,$$

$$S^{\mu u} = 2(\mu_x u_y - \mu_y u_x), \qquad S^{\mu v} = 2(\mu_x v_y - \mu_y v_x)\,.$$

FINITE ELEMENT FORMULATION

Weighted residuals equations

The weighted residual form of the governing equations is obtained by multiplying each by a weight function W and integrating over the domain:

$$\iint_A W_{1i}\left(\nabla^2 u + \omega_y + \left(\frac{u\rho_x + v\rho_y}{\rho}\right)_x\right)\mathrm{d}A = 0, \tag{7}$$

$$\iint_A W_{2i}\left(\nabla^2 v + \omega_x + \left(\frac{u\rho_x + v\rho_y}{\rho}\right)_y\right)\mathrm{d}A = 0, \tag{8}$$

$$\iint_A W_{3i}(\nabla^2(\mu\omega) - Re(\rho u\omega_x + \rho v\omega_y + S^\rho) + (S^{\mu u})_x + (S^{\mu v})_y)\mathrm{d}A = 0\,. \tag{9}$$

The weak form of these equations results after integration by parts of the second-order terms:

$$\iint_A\left[(W_{1i})_x u_x + (W_{1i})_y u_y - W_{1i}\omega_y + (W_{1i})_x\left(\frac{u\rho_x + v\rho_y}{\rho}\right)\right]\mathrm{d}A - \oint_C W_{1i}(-u_y\mathrm{d}x - v_y\mathrm{d}y) = \mathrm{Res}_{ui}, \tag{10}$$

$$\iint_A\left[(W_{2i})_x v_x + (W_{2i})_y v_y + W_{2i}\omega_x + (W_{2i})_y\left(\frac{u\rho_x + v\rho_y}{\rho}\right)\right]\mathrm{d}A - \oint_C W_{2i}(v_x\mathrm{d}y + u_x\mathrm{d}x) = \mathrm{Res}_{vi}, \tag{11}$$

$$\iint_A\Big[(W_{3i})_x(\mu\omega)_x + (W_{3i})_y(\mu\omega)_y + W_{3i}Re(\rho u\omega_x + \rho v\omega_y + S^\rho) + (W_{3i})_x(S^{\mu u})_x$$
$$+ (W_{3i})_y(S^{\mu v})_y\Big]\mathrm{d}A - \oint_C W_{3i}((\mu\omega)_n\mathrm{d}s + S^{\mu u}\mathrm{d}y - S^{\mu v}\mathrm{d}x) = \mathrm{Res}_{\omega i}, \tag{12}$$

where Res_{ui}, Res_{vi} and $\mathrm{Res}_{\omega i}$ are the residuals, using values from the previous iteration. The contour integrals of equations (10) and (11) are derived in detail in the boundary conditions section.

Newton–Galerkin form of the equations

The velocity–vorticity equations are solved simultaneously by a Newton method to achieve very rapid convergence. The linearized versions of equations (10), (11) and (12) are, with second-order terms neglected, as follows:

$$\sum_{j=1}^{8} K_{1ij}\Delta u_j + \sum_{j=1}^{8} K_{2ij}\Delta v_j + \sum_{j=1}^{4} K_{3ij}\Delta\omega_j = -\mathrm{Res}_{ui}, \tag{13}$$

where

$$K_{1ij} = \iint_A (W_{1i})_x\left((N_j^u)_x + \frac{\rho_x}{\rho}N_j^u\right) + (W_{1i})_y((N_j^u)_y)\mathrm{d}A\,,$$

$$K_{2ij} = \iint_A (W_{1i})_x\left(\frac{\rho_y}{\rho}N_j^u\right)\mathrm{d}A\,,$$

$$K_{3ij} = \iint_A -W_{1i}((N_j^\omega)_y)\mathrm{d}A\,;$$

$$\sum_{j=1}^{8} K_{4ij}\Delta v_j + \sum_{j=1}^{8} K_{5ij}\Delta u_j + \sum_{j=1}^{4} K_{6ij}\Delta\omega_j = -\mathrm{Res}_{vi}, \tag{14}$$

where

$$K_{4ij} = \iint_A (W_{2i})_y\left((N_j^u)_y + \frac{\rho_y}{\rho}N_j^u\right) + (W_{2i})_x((N_j^u)_x)\mathrm{d}A\,,$$

$$K_{5ij} = \iint_A (W_{2i})_y\left(\frac{\rho_x}{\rho}N_j^u\right)\mathrm{d}A\,,$$

$$K_{6ij} = \iint_A -W_{2i}((N_j^\omega)_x)\,\mathrm{d}A\,;$$

$$\sum_{j=1}^{4} K_{7ij}\Delta\omega_j + \sum_{j=1}^{8} K_{8ij}\Delta u_j + \sum_{j=1}^{8} K_{9ij}\Delta v_j = -\mathrm{Res}_{\omega i}\,, \tag{15}$$

where

$$K_{7ij} = \iint_A (W_{3i})_x(\mu_x N_j^\omega + \mu(N_j^\omega)_x) + (W_{3i})_y(\mu_y N_j^\omega + \mu(N_j^\omega)_y)$$
$$+ W_{3i}\,Re\,\rho(u(N_j^\omega)_x + v(N_j^\omega)_y)\,\mathrm{d}A\,,$$

$$K_{8ij} = \iint_A W_{3i}\,Re\left[\rho\omega_x N_j^u + \rho_x(u_y N_j^u + u(N_j^u)_y) - \rho_y(u_x N_j^u + u(N_j^u)_x)\right]\mathrm{d}A\,,$$

$$K_{9ij} = \iint_A W_{3i}\,Re\left[\rho\omega_y N_j^u + \rho_x(v_y N_j^u + v(N_j^u)_y) - \rho_y(v_x N_j^u + v(N_j^u)_x)\right]\mathrm{d}A\,.$$

Here Δu, Δv and $\Delta\omega$ are the changes in the x-velocity, y-velocity and vorticity respectively between iterations.

The weight functions W in the Galerkin weighted residual scheme are chosen to be the corresponding shape functions to each of the variables:

$$W_{1i} = W_{2i} = N_i^u, \qquad W_{3i} = N_i^\omega\,.$$

For equal degree of approximation of vorticity and derivatives of velocities, the former is represented by bilinear shape functions while the latter are approximated by biquadratic shape functions:

$$u = \sum_{j=1}^{8} N_j^u u_j, \qquad v = \sum_{j=1}^{8} N_j^u v_j, \qquad \omega = \sum_{j=1}^{4} N_j^\omega \omega_j\,.$$

The geometry is represented by 8-node curvilinear elements.

BOUNDARY CONDITIONS

Inlet

By specifying the velocity distribution at the inlet, the vorticity is also known and becomes a Dirichlet boundary condition at inlet nodes.

Exit

As noted in the derivation of the velocity equations, only the gradient of the continuity equation is set to zero and the continuity equation condition must therefore be explicitly imposed, at least at one point, to remove the arbitrariness of the solution.[12] In the present work this is done by using the continuity equation to modify the original contour integrals of equations (10) and (11). For example, in equation (10) the contour integral

$$\oint_C W_{1i}\left(u_x\,\mathrm{d}y - u_y\,\mathrm{d}x + \left(\frac{u\rho_x + v\rho_y}{\rho}\right)\mathrm{d}y\right)$$

has its compressible term modified using the continuity equation to yield

$$\oint_C W_{1i}\left(u_x \mathrm{d}y - u_y \mathrm{d}x + \left(\frac{u_x\rho + v_y\rho}{\rho}\right)\mathrm{d}y\right) = \oint_C W_{1i}\left(- u_y \mathrm{d}x - v_y \mathrm{d}y\right).$$

The contour integral of the y-velocity (equation 11)) is modified in a similar manner to yield

$$\oint_C W_{2i}(v_x \mathrm{d}y + u_x \mathrm{d}x).$$

Therefore, for an exit aligned with the y-axis, i.e. $\mathrm{d}x = 0$, the transverse velocity boundary condition ($v_x = 0$) is accounted for by dropping the contour integral of the y-velocity equation (11). The continuity equation is satisfied at the exit by evaluating the contour integral of the x-velocity equation (10), $-v_y \mathrm{d}y$. It is important to note that this term must be included in the Jacobian operating matrix of the Newton method.

The vorticity boundary condition is accounted for by dropping the contour integral of equation (12), since on the exit boundary the normal derivative of the vorticity is assumed to be zero.

Walls

At walls, no-slip and no-penetration are implemented as Dirichlet boundary conditions on the velocity components. The vorticity, however, has no explicit wall boundary condition and its definition is used in the following manner at the walls:

$$\iint_A N_i^{\omega}(\omega + u_y - v_x)\mathrm{d}A = 0. \tag{16}$$

UPDATING THE PRESSURE, DENSITY AND VISCOSITY

Pressure

The pressure in the incompressible formulation is not needed during the iteration process and is thus only solved for as *a posteriori* information. For subsonic flow, however, the pressure is needed to calculate the density and hence is an integral part of the iteration process. A Poisson equation whose natural boundary conditions automatically satisfy the normal momentum equation can be used to solve for the pressure.[3,4] It is derived by taking the divergence of the momentum equations:

$$\nabla\cdot(\nabla p + \mathbf{F}) = 0, \tag{17}$$

where F_x is the remainder of the x-momentum equation (2) and F_y is the remainder of the y-momentum equation (3). The weighted residual form of the pressure equation is

$$\iint_A W_i(\nabla\cdot(\nabla p + \mathbf{F}))\mathrm{d}A = 0, \tag{18}$$

Upon integration by parts,

$$\iint_A (\nabla W_i\cdot(\nabla p + \mathbf{F}))\mathrm{d}A = \oint_C W_i(\nabla p + \mathbf{F})\cdot\mathbf{n}\,\mathrm{d}s. \tag{19}$$

The contour integral contains the normal momentum equation and is simply set to zero on boundaries where the pressure is unknown.

Density

The density is obtained from the energy equation once the pressure is known. In the present work a constant total enthalpy is assumed:

$$\frac{\gamma}{\gamma-1}\frac{p}{\rho}+\tfrac{1}{2}(u^2+v^2)=H_\infty. \tag{20}$$

This is a good approximation in the absence of heat transfer. The complete energy equation can be solved for more general cases if needed.

Viscosity

A variable viscosity is obtained using Sutherland's empirical law for air:[17]

$$\frac{\mu}{\mu_\infty}=\left(\frac{T}{T_\infty}\right)^{3/2}\left(\frac{T_\infty+110\ \mathrm{K}}{T+110\ \mathrm{K}}\right). \tag{21}$$

The temperature is evaluated from the pressure and density using the equation of state for a perfect gas.

SOLUTION METHODOLOGY

The solution system consists of seven equations: vorticity transport, two velocity equations, pressure, energy, viscosity–temperature and an equation of state. The unknowns are the two velocities, the vorticity, the density, the pressure, the viscosity and the temperature.

The coupling of the density and viscosity with the velocity–vorticity equations for subsonic flows is not high enough to warrant the velocity–vorticity equations to be solved simultaneously with the remaining four equations. The coupling of the velocity and vorticity through the convective term is, however, very significant and the vorticity and velocities should therefore be solved simultaneously. The pressure is solved for from equation (19) with the updated velocities and vorticity. The density is calculated using the energy equation and the temperature and viscosity are then updated.

To get within the radius of convergence of the Newton method, marching in Reynolds number may be necessary, i.e. successive solutions at increasing *Re* may have to be obtained. In these cases incompressible Stokes flow, $Re = 0$, can be used as the initial guess and intermediate-*Re* solutions do not have to be fully converged before stepping up *Re*.[4] In the present work stepping up the *Re* was not found necessary and all solutions were obtained at the target *Re* directly.

COMPUTATIONAL RESULTS

Results have been obtained for laminar two-dimensional incompressible and subsonic flows. The cases investigated are flow in a driven cavity, over a trough and in a nozzle.

First the driven cavity problem is tested for incompressible flow at $Re = 400$. Figure 1 shows that convergence to an L-2 residual of 10^{-8} is attained in 11 iterations on a (15×15)-element grid. The initial guess needs about five iterations to adjust, owing to the corner separation zones, before quadratic convergence is attained. Figure 2 details these velocity vectors near the corner

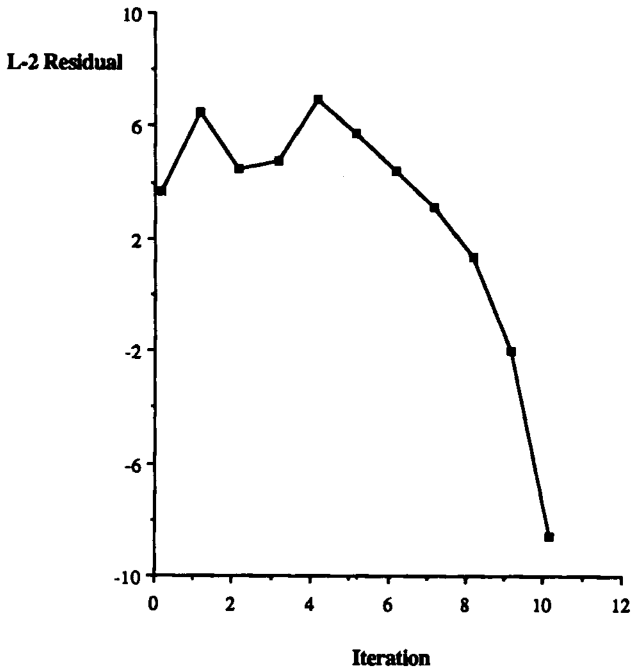

Figure 1. Convergence history for flow in a driven cavity, $Re = 400$

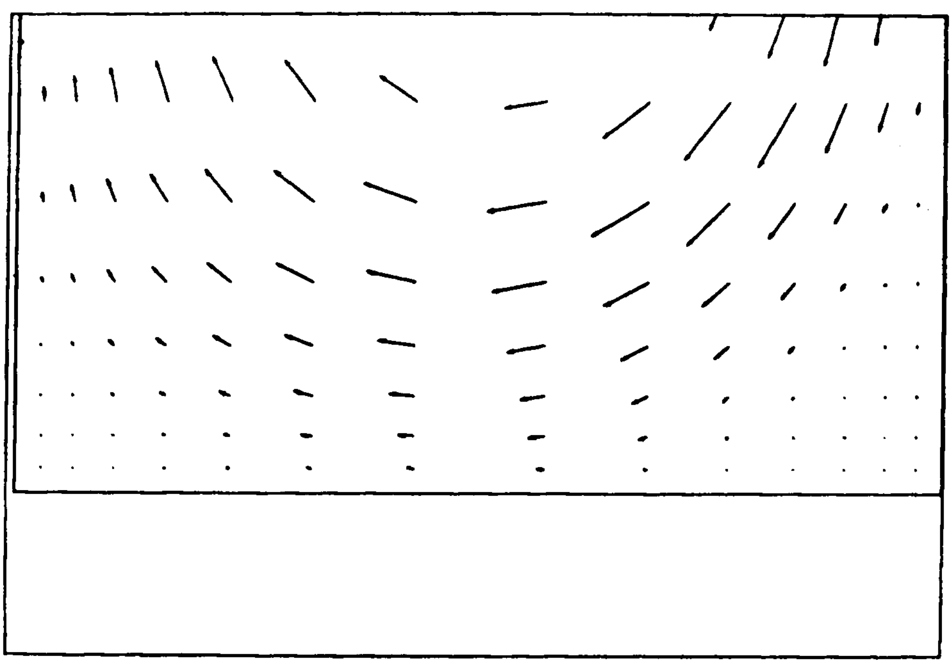

Figure 2. Velocity vectors for driven cavity, $Re = 400$

separation zones. Figures 3 and 4 compare equivorticity contours and centreline velocities respectively with the results of a streamfunction–vorticity solution.[3]

Flow over a trough is selected to test the stability of the method for high-Reynolds-number separated flows. The flow is fully developed at the inlet and a separation zone is formed and contained within the trough. For a Reynolds number of 10000, convergence is attained in six iterations to an L-2 residual of 10^{-8} on a 420-element grid and is shown in Figure 5. The initial guess is close enough to the solution to achieve quadratic convergence. The velocity vectors are shown in the region of the trough in Figure 6. The results are meant to be only illustrative since at such a Reynolds number the flow would actually be turbulent.

For subsonic flow a converging–diverging nozzle with an inlet Mach number of 0·2 and a Reynolds number of 100 is tested. Convergence is attained in 12 iterations to an L-2 residual of

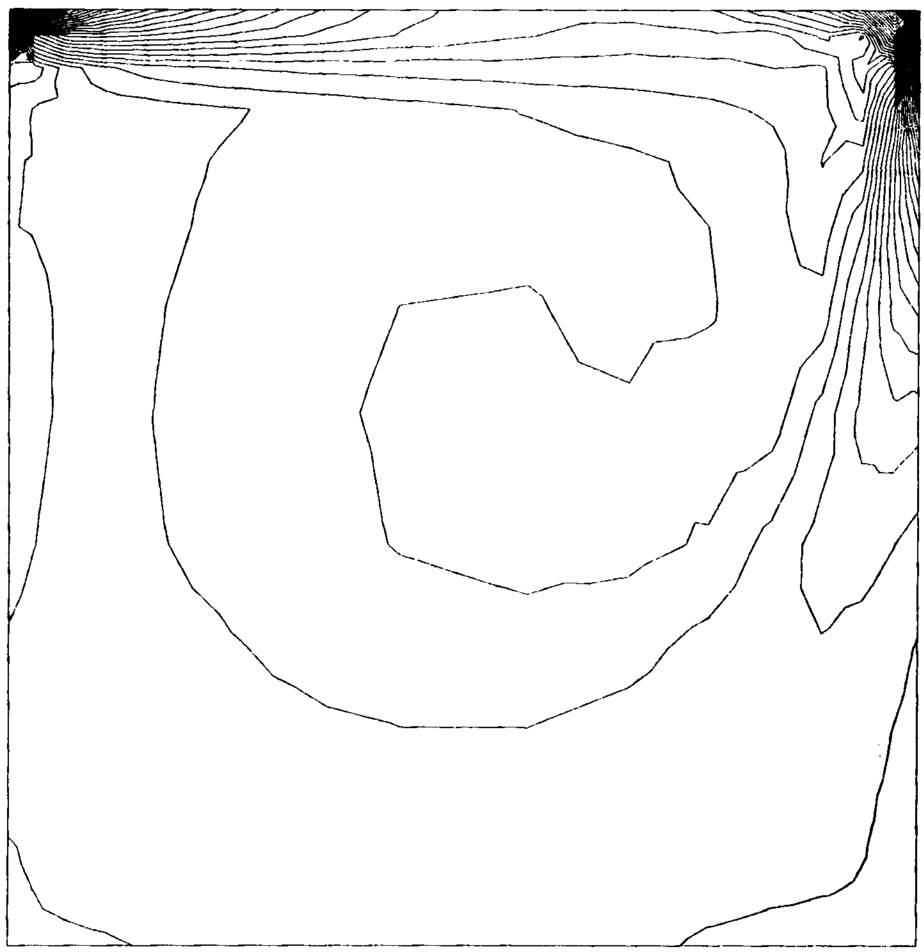

Figure 3(a). Equivorticity lines for driven cavity, velocity–vorticity, $Re = 400$

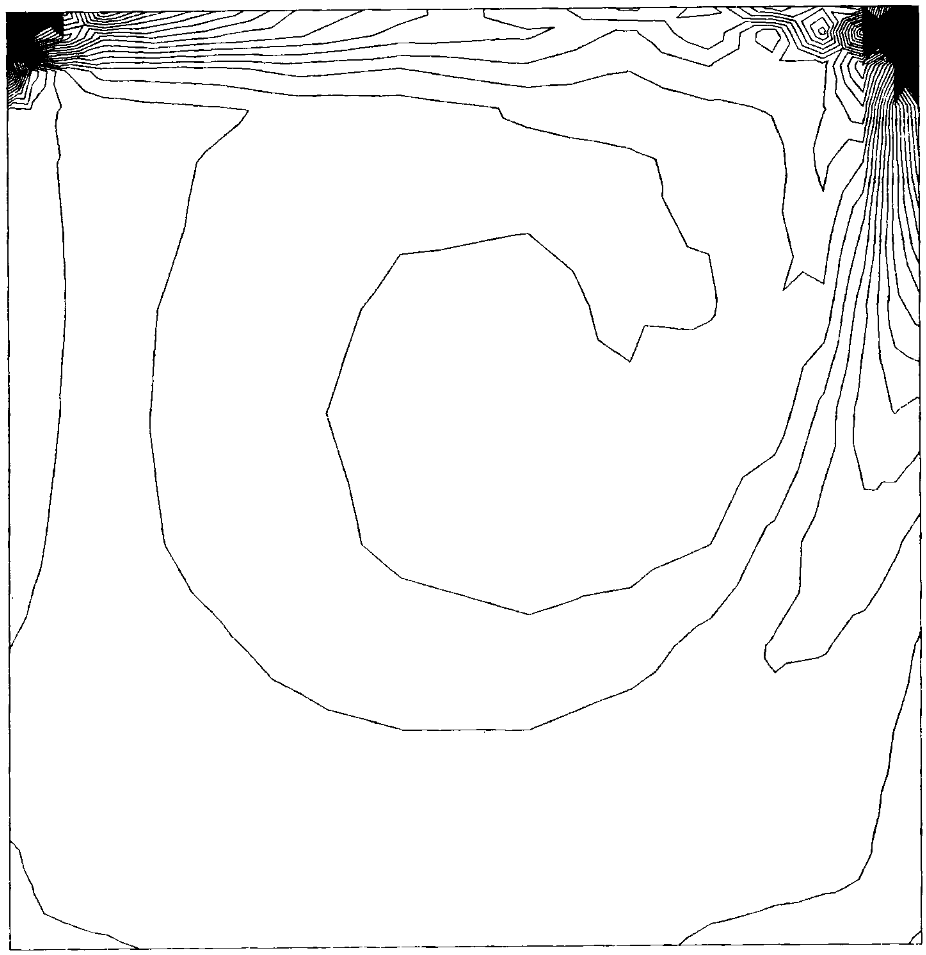

Figure 3(b). Equivorticity lines for driven cavity, streamfunction–vorticity, $Re = 400$

10^{-8} on a 930-element grid and is compared in Figure 7 to the convergence of the incompressible case with the same initial guess. As expected, convergence is quadratic for the incompressible case and drops to linear for the subsonic case, since density is lagged iteration-wise. Despite this, the convergence is quite fast, justifying the solution strategy chosen for subsonic flows, at least for this test case. Figure 8(a) shows the streamlines in the nozzle region for incompressible flow and Figure 8(b) shows details in the recirculation zone after the nozzle. Figure 9 compares the incompressible and subsonic results. The maximum Mach number in the subsonic flow case was 0·5 and the results indicate that the levels of velocity and vorticity are higher than the corresponding incompressible case.

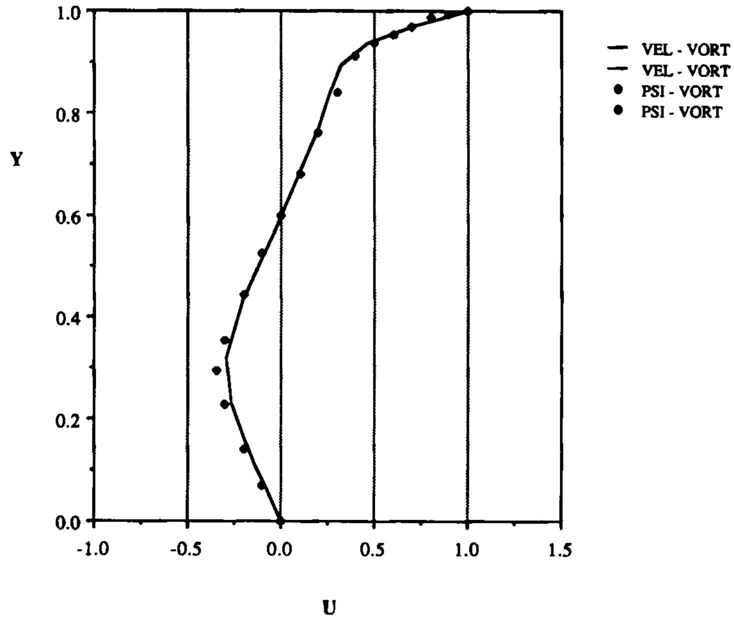

Figure 4. Centreline velocities for driven cavity, $Re = 400$

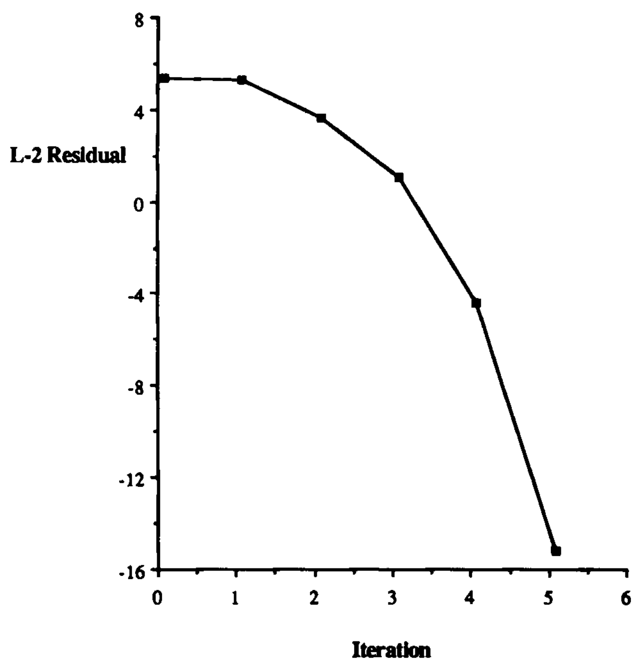

Figure 5. Convergence history for flow in a trough, $Re = 10000$

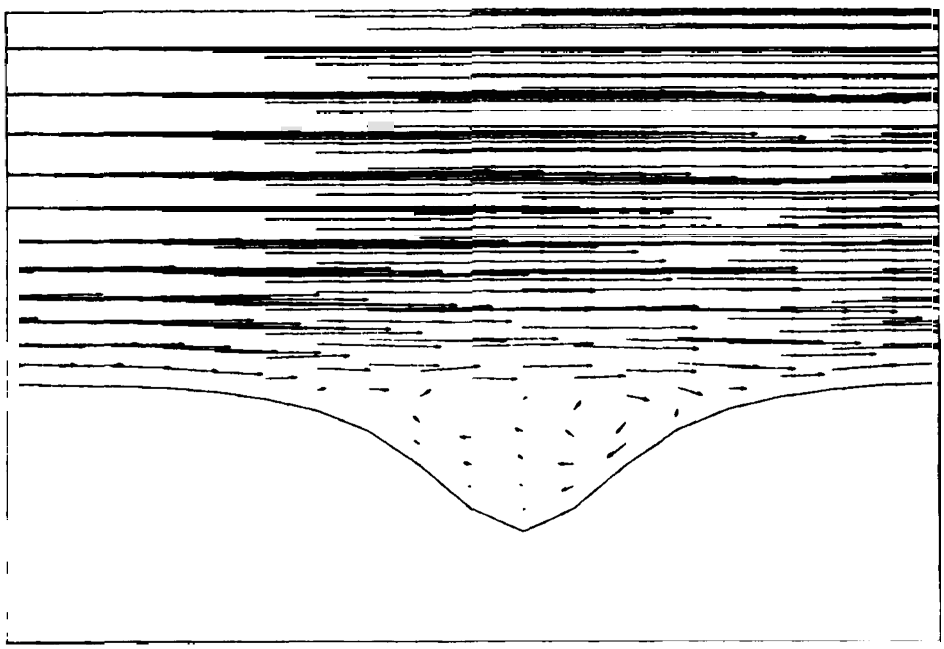

Figure 6. Velocity vectors for trough, $Re = 10000$

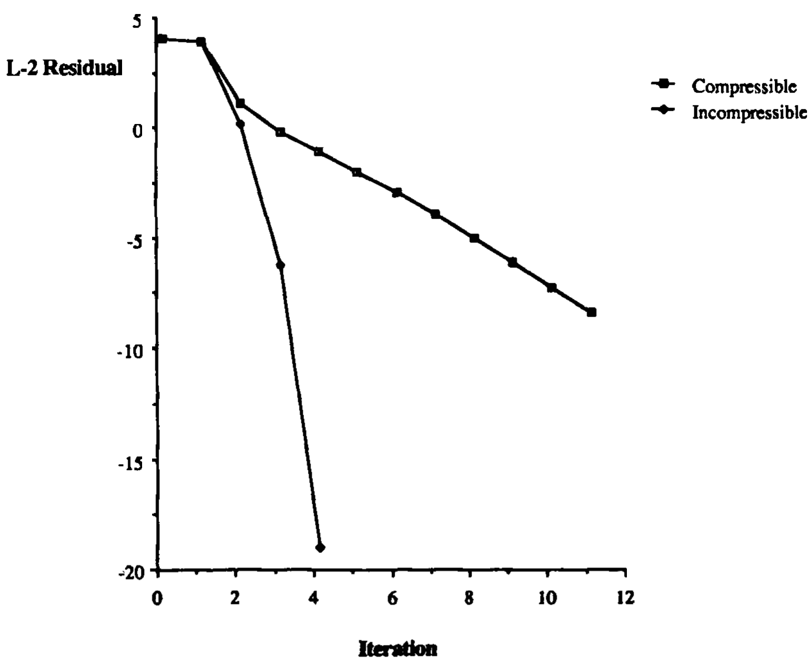

Figure 7. Convergence history for flow in a nozzle, $Re = 100$

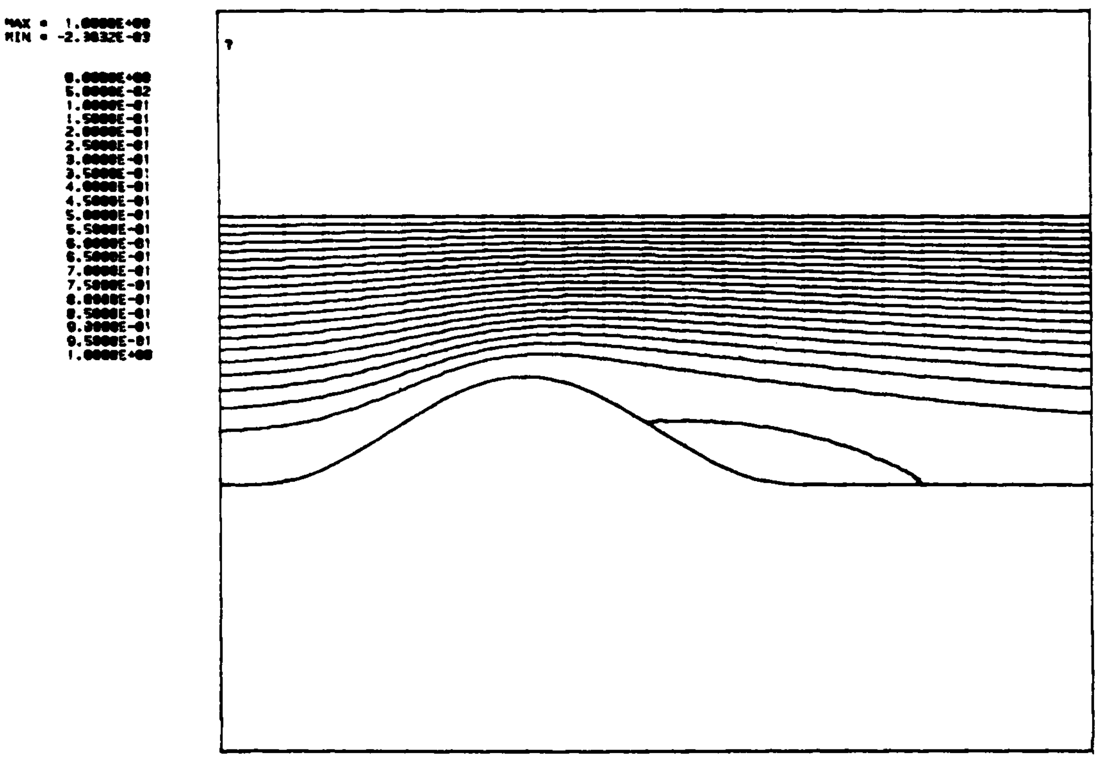

Figure 8(a). Streamlines for nozzle, $Re = 100$

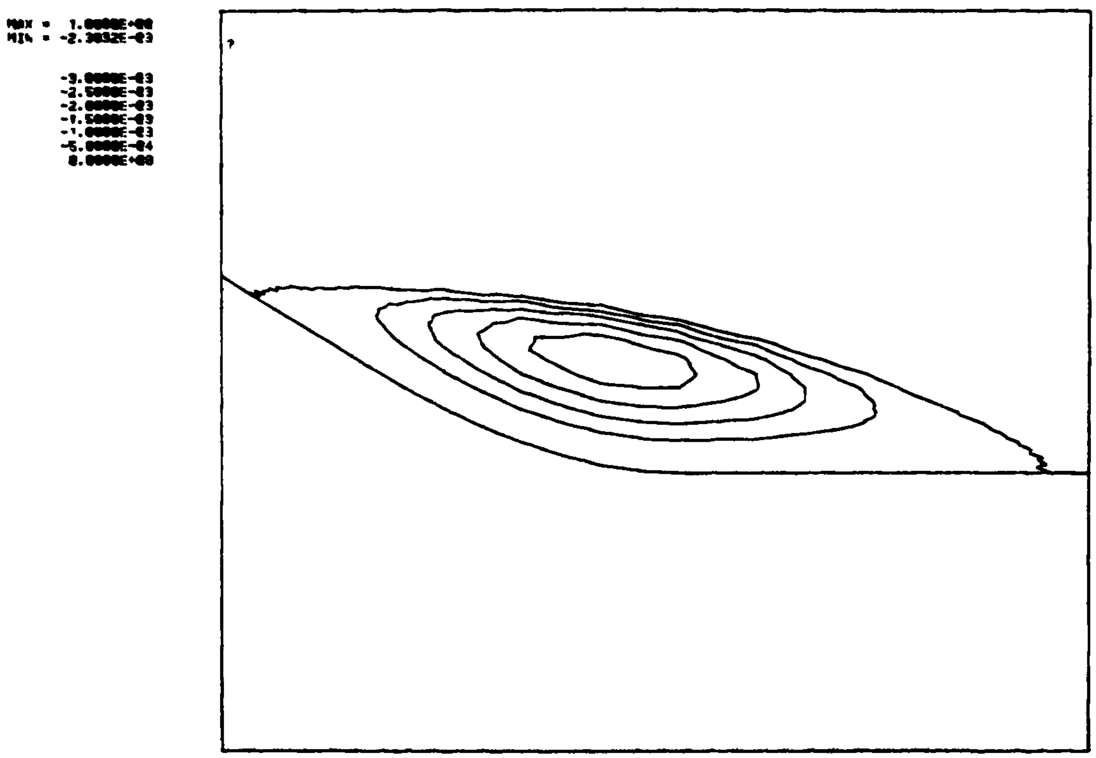

Figure 8(b). Streamline details in nozzle recirculation zone, $Re = 100$

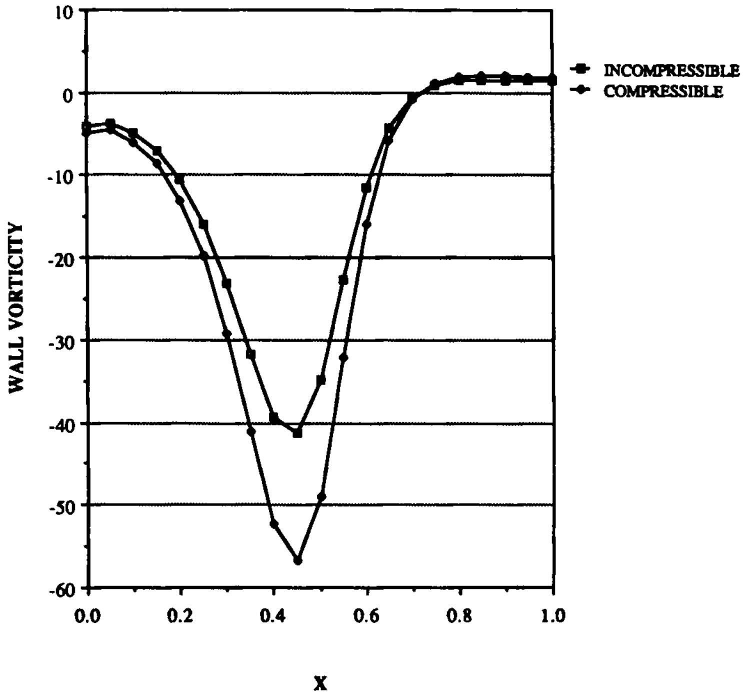

Figure 9. Wall vorticity profiles for nozzle, $Re = 100$

CONCLUSIONS

A finite element method has been presented for the velocity–vorticity formulation of incompressible and subsonic viscous flows. A finite element implementation of the wall vorticity boundary condition has been demonstrated and an accurate method of calculation for the pressure has also been shown. No artificial viscosity is needed to stabilize the iteration or smooth the solution. The imposition of boundary conditions is improved through the geometric accuracy of quadratic elements in approximating curved boundaries in practical applications.

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